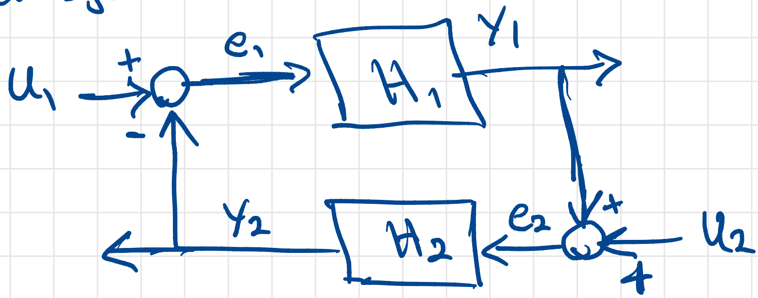
Small gain thm:

- Feedback sys.



- Assume the feedback sys is well-defined:

for every input $u_1, u_2 \in L_e$, there exists well-defined output $y_1, y_2 \in L_e$

- Overall system

$$\text{input: } u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \text{output: } y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

- Assume H_1 and H_2 are L-stable with finite-gain

$$\|y_1\|_L \le \gamma_1 \|e_1\|_L + \beta_1$$
$$\|y_2\|_L \le \gamma_2 \|e_2\|_L + \beta_2$$

- Question: is the overall sys L-stable?

Small-gain thm:

- Feedback connection is finite-gain L-stable if $\gamma_1\gamma_2 < 1$.

proof:

$$e_1 = u_1 - y_2$$
$$e_2 = u_2 + y_1$$

$$\Rightarrow \quad \|e_1\|_L \le \|u_1\|_L + \|y_2\|_L$$
$$\le \|u_1\|_L + \gamma_2\|e_2\|_L + \beta_2$$

Similarly

$$\|e_2\|_L \le \|u_2\|_L + \|y_1\|_L$$
$$\le \|u_2\|_L + \gamma_1\|e_1\|_L + \beta_1$$

$$\Rightarrow \quad \|e_1\|_L \le \|u_1\|_L + \gamma_2\|u_2\|_L + \gamma_1\gamma_2\|e_1\|_L + \beta_2 + \gamma_2\beta_1$$

$$\xRightarrow{\gamma_1\gamma_2<1} \quad \|e_1\|_L \le \frac{1}{1-\gamma_1\gamma_2}\Big(\underbrace{\|u_1\|_L + \gamma_2\|u_2\|_L}_{\le(1+\gamma_2)\|u\|_L} + \beta_2 + \gamma_2\beta_1\Big)$$

Similarly

$$\|e_2\| \leq \frac{1}{1-\gamma_1\gamma_2}\left(\underbrace{\|u_2\| + \gamma_1\|u_1\|}_{\leq (1+\gamma_1)\|u\|_L} + \beta_1 + \gamma_1\beta_2\right)$$

$$\Longrightarrow \quad \|e\|_L \leq \|e_1\|_L + \|e_2\|_L$$

$$\leq \frac{2+\gamma_1+\gamma_2}{1-\gamma_1\gamma_2}\|u\|_L + \beta$$

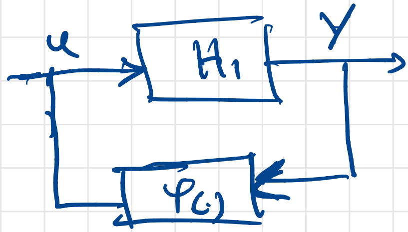
$$\beta = \frac{(1+\gamma_2)\beta_1 + (1+\gamma_1)\beta_2}{1-\gamma_1\gamma_2}$$

$$\Longrightarrow \quad \|y\|_L \leq \|e\|_L + \|u\|_L$$

$$\leq \left(\frac{2+\gamma_1+\gamma_2}{1-\gamma_1\gamma_2}+1\right)\|u\|_L + \beta$$

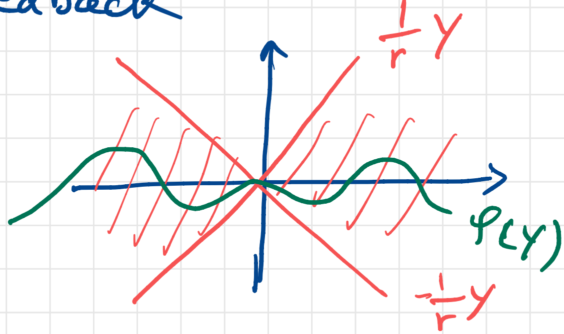
$\Longrightarrow$ L-stable with finite gain.

Example:



- u, y are scalar
- H_1 is L-stable with gain r
- $\varphi_2(\cdot)$ is static feedback (memoryless func.)

$$u = \varphi(y)$$



- By small-gain thm, the overall sys is L-stable if φ has gain smaller than $\frac{1}{r}$

$$\Rightarrow \quad |\varphi(y)| < \frac{1}{r}|y|$$

$\rightarrow$ sector condition!

Passivity:

$$\underbrace{\text{stored energy} \leq \overbrace{\text{energy inflow}}}_{\text{dissipative system.}}$$

Example: (pendelum with torque)

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a\sin(x_1) - bx_2 + u$$
$$y = x_2$$

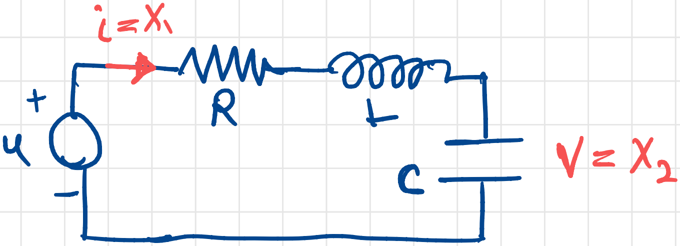
energy $\quad V(x) = a(1-\cos(x_1)) + \frac{1}{2}x_2^2$

$$\Longrightarrow \quad \dot{V}(x) = -bx_2^2 + x_2 u$$
$$= \underbrace{-bx_2^2}_{\text{dissipation}} + yu$$

$$\Longrightarrow \quad \underbrace{\dot{V}(x)}_{\text{rate of energy stored}} \leq \underbrace{yu}_{\text{rate of energy input}} \longrightarrow \text{Passive system}$$

torque × angular velocity

Example : (electrical circuit)



$$C \frac{dx_2}{dt} = x_1$$

$$L \frac{dx_1}{dt} + R x_1 + x_2 = u$$

$$y = x_1 \implies \text{energy inflow} = uy$$

energy $\quad V(x) = \frac{1}{2} C x_2^2 + \frac{1}{2} L x_1^2$

$$\Rightarrow \dot{V}(x) = C x_2 \dot{x}_2 + L x_1 \dot{x}_1$$

$$= -R x_1^2 + uy$$

$$\Rightarrow \dot{V}(x) \leq uy \longrightarrow \text{Passive}$$

Definition :

The system

$$\dot{x} = f(x,u)$$
$$y = h(x,u)$$

storage function

is passive if $\exists$ a function $V(x)$ s.t. $\underbrace{V(x) \geq 0}_{\text{positive semi-definite}}$

and

$$\underbrace{\dot{V}(x)}_{\frac{\partial V}{\partial x} f(x,u)} \leq u^T y \qquad \forall x,u$$

- Moreover, it is said to be

① lossless if

$$\dot{V} = u^T y$$

② input strictly passive if

$$\dot{V} \leq u^T y - u^T \varphi(u) \quad \text{and} \quad u^T \varphi(u) > 0 \quad \forall u \neq 0$$

③ output strictly passive if

$$\dot{V} \leq u^T y - y^T \varphi(y) \quad \text{and} \quad y^T \varphi(y) > 0 \quad \forall y \neq 0$$

④ strictly passive if

$$\dot{V} \leq u^T y - W(x) \quad \text{and} \quad W(x) > 0 \quad \forall x \neq 0$$

Example:

① Pendelum

$$\dot{V} = uy - b x_2^2$$
$$= uy - by^2$$

→ output strictly passive

if $b = 0 \Rightarrow$ lossless

②
$$\dot{x} = -ax + u$$
$$y = x$$

take $V(x) = \frac{1}{2}x^2$

$$\Rightarrow \dot{V}(x) = -ax^2 + uy$$

strictly passive

also output strictly passive

③ memoryless functions (no state)

$$y = h(u)$$

- in this case $V = 0$

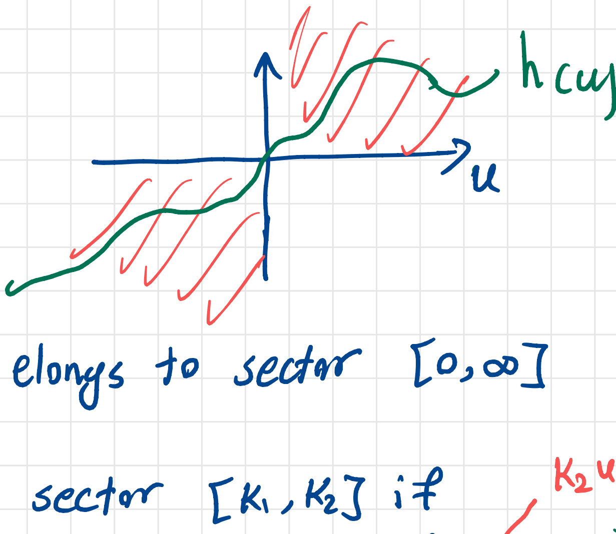
- passive if $u^T y \geq 0$

$$\Updownarrow$$

$$u^T h(u) \geq 0$$

- if u and y are scalar

$$u h(u) \geq 0$$



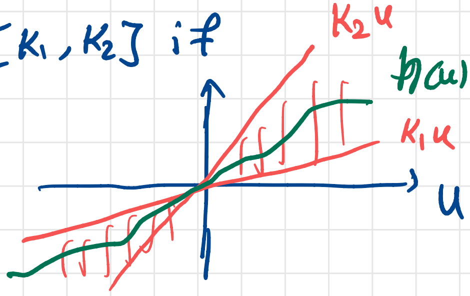
- we say h belongs to sector $[0, \infty]$

more generally,

- h belongs to sector $[K_1, K_2]$ if

$$K_1 u^2 \leq u h(u) \leq K_2 u^2$$

$$\Longleftrightarrow (h(u) - K_1 u)(h(u) - K_2 u) \leq 0$$



- if $h \in [K_1, \infty]$, then input strict passive

$$uy = uh(u) \geq K_1 u^2$$

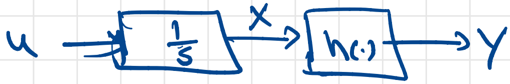
- if $h \in [0, K_2]$, then output strict passive

$$uy = uh(u) \geq \frac{1}{K_2} h^2(u) = \frac{1}{K_2} y^2$$

- The sector definition can be extended to vector case (see pp 232)

Example:

a)
$$\dot{x} = u$$
$$y = h(x)$$

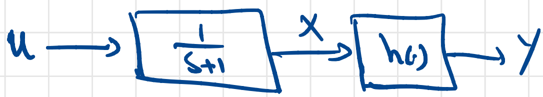


where $h \in [0, \infty]$

- let $V(x) = \int_0^x h(z)\,dz$

$$\Rightarrow \dot{V}(x) = h(x)\,u = yu \rightarrow \text{lossless}$$

b)
$$\dot{x} = -x + u$$
$$y = h(x)$$



$$V(x) = \int_0^x h(z)\,dz$$

$$\Rightarrow \dot{V}(x) = h(x)\,u - x\,h(x)$$
$$= yu - x\,h(x) \rightarrow \text{strictly passive}$$